REGLAS FUNDAMENTALES DE DIFERENCIACION

POR

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(Continuacion)

11. $$d \operatorname{arc\,tg} \frac{u+v}{1-uv} = d\,(\operatorname{arctg} u + \operatorname{arctg} v) = \frac{du}{1+u^2} + \frac{dv}{1+v^2}$$

12. $$d \operatorname{arc\,sen} 2\,x\sqrt{1-x^2} = 2\frac{d\,x\sqrt{1-x^2}}{\sqrt{1-4\,x^2\,(-x^2)}} = \frac{2\,dx}{\sqrt{1-x^2}}$$

En efecto, sea $x = \operatorname{sen} a \;\therefore\; \cos a = \sqrt{1-x^2}$,

$$2 \operatorname{sen} a \cos a = \operatorname{sen} 2\,a = 2\,x\sqrt{1-x^2} \;\therefore\; 2\,a = y.$$

13. $$d \operatorname{arc\,tg} \frac{e^x - e^{-x}}{e^x + e^{-x}} = d \operatorname{arctg} \frac{e^{2\,x} - 1}{e^{2\,x} + 1}$$

$$= d\,\frac{e^{2x}-1}{e^{2x}-1} : \left[1+\left(\frac{e^{2x}-1}{e^{2x}+1}\right)\right]$$

$$= \frac{(e^{2x}+1)\,e^{2x}.\,2-(e^{2x}-1)\,e^{2x}.\,2}{(e^{2x}+1)^2+(e^{2x}-1)^2} = \frac{2}{e^{2x}+e^{-2x}}\,dx.$$

14. $$d\,\text{arc cos}\,\frac{b+a\cos x}{a+b\cos x}$$ (Williamson, 24)

$$= -\frac{(b^2-a^2)\,\text{sen}\,x\,d\,x}{(a+b\cos x)^2} : \sqrt{1-\left(\frac{b+a\cos x}{a+b\cos a}\right)^2}$$

$$= \frac{(a^2-b^2)\,\text{sen}\,x\,d\,x}{(a+b\cos x)\sqrt{(a+b\cos x)^2-(b+a\cos x)^2}} = \frac{\sqrt{a^2-b^2}}{a+b\cos x}\,dx$$

15. $$y = L\sqrt{\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}} = \tfrac{1}{2}\,L\,\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$= \tfrac{1}{2}\,L\,\frac{(\sqrt{1+x}+\sqrt{1-x})^2}{2\,x}$$

$$= \tfrac{1}{2}\,L\,\frac{1+\sqrt{1-x^2}}{x} = \tfrac{1}{2}\,L\,(1+\sqrt{1-x^2})-\tfrac{1}{2}\,L\,x$$

$$d\,y = \tfrac{1}{2}\left(\frac{-x}{\sqrt{1-x^2}} : (1+\sqrt{1-x^2}) - \frac{1}{x}\right) d\,x = \frac{1}{2\,x\sqrt{1-x^2}}\,d\,x$$

16. $$y = \text{arc tg}\,\frac{\sqrt{1+x^2}-1}{x} + \text{arc tg}\,\frac{2\,x}{1-x^2}\,.$$

Hacemos $x = \operatorname{tg} z \quad \therefore \quad 1 + x^2 = 1 + \operatorname{tg}^2 z = \sec^2 z$

$$y = \operatorname{arc\,tg} \frac{\sec z - 1}{\operatorname{tg} z} + \operatorname{arc\,tg} \frac{2 \operatorname{tg} z}{1 - \operatorname{tg}^2 z}$$

$$\frac{\sec z - 1}{\operatorname{tg} z} = \frac{1 - \cos z}{\operatorname{sen} z} = \frac{2 \operatorname{sen}^2 \frac{1}{2} z}{2 \operatorname{sen} \frac{1}{2} z \cos \frac{1}{2} z} = \operatorname{tg} \tfrac{1}{2} z;$$

$$\frac{2 \operatorname{tg} z}{1 - \operatorname{tg}^2 z} = \operatorname{tg} 2z; \; y = \operatorname{arc\,tg} (\operatorname{tg} \tfrac{1}{2} z) + \operatorname{arc\,tg} (\operatorname{tg} 2 z)$$

$$dy = \frac{\sec^2 \frac{1}{2} z . \frac{1}{2} dz}{1 + \operatorname{tg}^2 \frac{1}{2} z} + \frac{\sec^2 z . 2 dz}{1 + \operatorname{tg}^2 2z}$$

$$= \tfrac{1}{2} dz + 2 dz = \tfrac{5}{2} dz = \tfrac{5}{2} d \operatorname{arc\,tg} x = \tfrac{5}{2} \frac{1}{1 + x^2} dx$$

17. $$y = \frac{a \cos bx + b \operatorname{sen} bx}{a^2 + b^2} e^{ax}$$ (Laurent, 224)

$$dy = \frac{1}{a^2 + b^2} [(a \cos b x + b \operatorname{sen} bx) e^{ax} a$$

$$+ e^{ax} (-ab \operatorname{sen} bx + b^2 \cos bx)]$$

$$= \frac{e^{ax}}{a^2 + b^2} (a^2 + b^2) \cos bx = e^{ax} \cos bx$$

18. $$y = \mathrm{L} (\mathrm{L} \operatorname{sen} x); \; dy = \frac{d \mathrm{L} \operatorname{sen} x}{\mathrm{L} \operatorname{sen} x} dx = \frac{\cot x}{\mathrm{L} \operatorname{sen} x} dx$$

19. $$d \operatorname{arc\,tg} e^x = \frac{e^x dx}{1 + e^{2x}} = \frac{1}{e^x + e^{-x}} dx$$

320. $$y = x^{\mathrm{L} x}; \; \mathrm{L}\, y = \mathrm{L} x . \mathrm{L} x = \mathrm{L}^2 x$$

$$\frac{dy}{y}=2\,\mathrm{L}\,x\,d\,\mathrm{L}\,x=2\,\mathrm{L}\,x\frac{dx}{x}\;.\cdot.\;\frac{dy}{dx}=x^{\,\mathrm{L}x}\frac{2\,\mathrm{L}\,x}{x}$$

1. $$y=(1+x^2)\,e^{\operatorname{arc\,tg} x}$$ (Catalán 123)

$$\mathrm{L}\,y=e^{\operatorname{arctg} x}\,\mathrm{L}\,(1+x^2);\;\frac{dy}{y}=\left[e^{\operatorname{arc\,tg} x}\frac{2x}{1+x^2}\qquad+\mathrm{L}\,(1+x^2)\right.$$

$$\left.e^{\operatorname{arc\,tg} x}\frac{1}{1+x^2}\right|dx$$

$$.\cdot.\;ay=(1+x^2)^{e^{\operatorname{arc\,tg} x}}\,e^{\operatorname{arc\,tg} x}\left(\frac{2x}{1+x^2}+\frac{\mathrm{L}\,(1+x^2)}{1+x^2}\right)$$

2. $y=\operatorname{arc\,sen}\left(x\sqrt{1-a^2}+a\sqrt{1-x^2}\right)$ (Niewenglowski, 73)

$=\operatorname{arc\,sen} x+\operatorname{arc\,sen} a$ (N.º 29, C, b)

$$dy=\frac{dx}{\sqrt{1-x^2}}$$

3. $y=e$ $\quad dy=e$ $\quad e$ . . . dx

4. $y=\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^x=u^x . \quad \therefore L\,y=x\,L\,u$

$$\frac{dy}{y}=x\frac{du}{u}+L\,u\,d\,x;\; d\,u=\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)dx$$

$$dy=\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^x\left[x\frac{-\frac{1}{x^2}-\frac{2}{x^3}}{1+\frac{1}{x}+\frac{{1}}{x^2}}+L\left(1+\frac{1}{x}+\frac{1}{x^2}\right)\right]dx$$

$$=\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^x\left[L\,(1+\frac{1}{x}+\frac{1}{x^2}-\frac{x+2}{x^2+x+1}\right]dx$$

5. $y=(x^2-1)\,e^{x^2}$ (Fabry, 7)

$$L\,y=L\,(x^2-1)\;\; x^2=L\,(x+1)+L\,(x-1)+x^2$$

$$\frac{dy}{y}=\left(\frac{1}{x+1}+\frac{1}{x-1}+2x\right)dx=\frac{2\,x^8}{x^2-1}\,dx$$

$$\therefore\; dy=2\,e^{x^2}\,x^3\,d\,x$$

6. $y=L\,(\sqrt{1+x^2}-\sqrt{1-x^2})=L\sqrt{2-2\sqrt{1-x^4}}$

$$=\tfrac{1}{2}\,L\,[\,2\,(1-\sqrt{1-x^4})\,]=\tfrac{1}{2}\,L\,2+\tfrac{1}{2}\,L\,(1-\sqrt{1-x^4})$$

$$dy=\tfrac{1}{2}\,\frac{4\,x^3\,d\,x}{(1-\sqrt{1-x^4})\,2\sqrt{1-x^4}}=\frac{1+\sqrt{1-x^4}}{x\sqrt{1-x^4}}\,d\,x$$

7. $u=\frac{1}{2\,i}\ \mathrm{L}\ (i-z)-\mathrm{L}\,(i+z)+2\,k\,\pi\,i\,]$ (JORDAN, 239)

$$du=\frac{1}{2\,i}\left(\frac{-1}{i-z}-\frac{1}{i+z}\right)dz=\frac{d\,z}{1+z^2}$$

8. $y=x^{\mathrm{L}x}$ (Laurent, 82)

Sea $u=\mathrm{L}\,x$; resulta la funcion reducida

$$y=x^u$$

$$\therefore\ \mathrm{L}\,y=u\,\mathrm{L}\,x \quad \text{y} \quad \frac{d\,y}{y}=\mathrm{L}\,x\,d\,u+u\,\frac{d\,x}{x}$$

$$\therefore\ dy=y\left(\mathrm{L}\,x\,\frac{d\,x}{x}+\mathrm{L}\,x\,\frac{d\,x}{x}\right)=y.\ 2\,\mathrm{L}\,x\,\frac{d\,x}{x}$$

$$d\,y=x.^{\mathrm{L}x}\,\mathrm{L}\,x^2\,\frac{d\,x}{x}=2\,x^{\mathrm{L}x-1}\,\mathrm{L}\,x\,d\,x.$$

9. $y=\mathrm{Ltg}\left(\frac{\pi}{4}+\frac{x}{2}\right)$ (Serret, 50)

Sea $u=\mathrm{tg}\,\varphi$, y $\varphi=\frac{\pi}{4}+\frac{x}{2}$, queda $y=\mathrm{L}u\ \therefore\ dy=\frac{d\,u}{u}$ (A)

$$d\,u = d\,(\mathrm{tg}\,v) = \sec^2 v\,d\,v,\ \frac{d\,u}{u} = \frac{\sec^2 v}{\mathrm{tg}\,v}\,d\,v = \frac{2}{\mathrm{sen}\,2\,v}\,d\,v \qquad \text{(B)}$$

$$d\,v = d\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2}\,dx \qquad \text{(C)}$$

Sustituyendo (B) y (C) en (A).

$$d\,y = \frac{2}{\mathrm{sen}\left(\frac{\pi}{2} + x\right)}\,\frac{1}{2}\,d\,x = \frac{d\,x}{\mathrm{sen}\left(\frac{\pi}{2} + x\right)}$$

$$\therefore\ d\left[\mathrm{L\,tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \frac{d\,x}{\cos x}$$

330. $y = (\cos x)^{\mathrm{sen}\,x}$ (Lacroix, 160)

Hacemos $\cos x = v$, $\mathrm{sen}\,x = u$, y queda

$$y = v^u \quad \therefore \quad d\,y = v^u\,\mathrm{L}\,v\,d\,u + v^{u-1}\,u\,d\,v$$

$$d\,u = d\,(\mathrm{sen}\,x) = \cos x\,d\,x,\ d\,v = d\,(\cos x) = -\mathrm{sen}\,x\,d\,x\ .$$

Sustituyendo, $d\,y = (\cos x)^{\mathrm{sen}\,x}\,\mathrm{L}\,\cos x\,\cos x\,d\,x - \cos x^{\mathrm{sen}\,x-1}\,\mathrm{sen}^2 x\,d\,x$ ó bien, $d\,y = (\cos x)^{\mathrm{sen}\,x}\,(\cos x\,\mathrm{L}\,\cos x - \mathrm{tg}\,x\,\mathrm{sen}\,x)\,d\,x$.

1. $$y=\text{sen}\,\frac{a\,x}{\sqrt{1-a^2\,x^2}}$$ (Navier, 26)

Sea $u=\dfrac{a\,x}{\sqrt{1-a^2\,x^2}}$ $\therefore$ $y=\text{sen}\,u$, y $d\,y=\cos\,u\,d\,u$

Para encontrar du, aplicamos L,

$$\text{L}u=\text{L}a+\text{L}x-\tfrac{1}{2}\,\text{L}\,(1-a^2\,x^2)$$

$$\therefore\ \frac{d\,u}{u}=\frac{d\,x}{x}-\frac{1}{2}\cdot\frac{2\,a^2\,x\,d\,x}{1-a^2\,x^2}$$

$$\therefore\ d\,u=\frac{a\,x}{(1-a^2\,x^2)^{\frac{1}{2}}}\left(\frac{1}{x}+\frac{a^2\,x}{1-a^2\,x^2}\right)d\,x=\frac{a}{(1-a^2\,x^2)^{\frac{3}{2}}}\,d\,x$$

$$\therefore\ d\,y=\frac{a}{(1-a^2\,x^2)^{\frac{3}{2}}}\cdot\cos\frac{a\,x}{\sqrt{1-a^2\,x^2}}\,d\,x$$

2. $$y=x^{\text{arc sen}\,x}$$ (Price, 69)

Aplicamos L a los dos miembros,

$$\text{L}\,y=\text{arc sen}\,x\,\text{L}\,x$$

$$\therefore\ \frac{d\,y}{y}=\text{L}\,x\frac{d\,x}{\sqrt{1-x^2}}+\text{arc sen}\,x\,\frac{d\,x}{x}$$

$$\therefore\ dy = x^{\text{arc sen } x}\left(\frac{x\,\mathrm{L}\,x + (1-x^{2})^{\frac{1}{2}}\,(\text{arc sen } x)}{x\ (1-x^{2})^{\frac{1}{2}}}\right)dx$$

3. $$y = x^{-\frac{1}{2}}\ \text{sen}\left\{\mathrm{L}\left(\frac{x}{a}\right)^{n} + \text{arc tg } 2\,n\right\}$$ (Greenhill, 62)

Hacemos $\mathrm{L}\left(\frac{x}{a}\right)^{n} + \text{arc tg } 2\,n = u \ \therefore\ y = x^{-\frac{1}{2}}\ \text{sen } u$

$$\therefore\ dy = x^{-\frac{1}{2}}\left(\cos u\ du - \frac{\text{sen } u}{2\,x}\,dx\right);\ du = \frac{n\left(\frac{x}{a}\right)^{n-1}\frac{1}{a}\,dx}{\left(\frac{x}{a}\right)^{n}} = \frac{n\,dx}{x}$$

$$\therefore\ dy = x^{-\frac{1}{2}}\left(\frac{n\cos u}{x}\,dx - \frac{\text{sen } u}{2\,x}\,dx\right) = \frac{1}{2}\,x^{-\frac{3}{2}}(2\,n\cos u - \text{sen } u)\,dx$$

Para simplificar, sea

$$2\,n = \text{tg}\,\varphi\ \therefore\ \varphi = \text{arc tg } 2n \quad \text{y} \quad \cos\varphi = \frac{1}{\sqrt{1+4\,n^{2}}}$$

$$\therefore\ 2\,n\cos u - \text{sen } u = \frac{1}{2}\left(\frac{\text{sen}\,\varphi\cos u}{\cos\varphi} - \text{sen } u\right)$$

$$= \frac{\text{sen}\,(\varphi - u)}{2\cos\varphi} = \frac{-\text{sen L}\left(\frac{x}{a}\right)^{n}}{\frac{1}{2\sqrt{1+4\,n^{2}}}}\ \therefore\ dy = x^{-\frac{3}{2}}\sqrt{u^{2}+\tfrac{1}{4}}\ \text{sen L}\left(\frac{x}{a}\right)^{n}$$

$$4.\ 2 \cos m x = (2 \cos x)^m - m (2 \cos x)^{m-2} + \frac{m(m-3)}{2\,!} (2 \cos x)^{m-4}$$

$$- \frac{m (m-4) (m-5)}{3\,!} (2 \cos x)^{m-6} + \dots \qquad \text{(Lardner, 26).}$$

Sea $2 \cos m x = y$, y $2 \cos x = u$

$$y = u^m - m\, u^{m-2} + \frac{m (m-3)}{2\,!} u^{m-4} - \dots$$

$$\therefore d y = \left(m\, u^{m-1} - m (m-2)\, u^{m-3} + \frac{m (m-3) (m-4)}{2\,!} \right.$$

$$\left. u^{m-5} - \dots \right) d u$$

$$d y = d (2 \cos m x) = 2\, d (\cos m x) = -2 \operatorname{sen} m x \,.\, m\, d x$$

$$d u = d (2 \cos x) = -2 \operatorname{sen} x\, d x$$

$$\therefore -2 m \operatorname{sen} m x\, d x = -[\, m (2 \cos x)^{m-1} - m (m-2)$$

$$(2 \cos x)^{m-3} + - \dots] \,.\, 2 \operatorname{sen} x\, d x$$

Simplificando y dividiendo por $d x$,

$$\operatorname{sen} m x = \operatorname{sen} x \left\{ (2 \cos x)^{m-1} - (m-2) (2 \cos x)^{m-3} + \right.$$

$$\frac{(m-3)\ (m-4)}{2\ !} \cdot (2 \cos x)^{m-5} \ldots \Big\}$$

5. $$y = \frac{1}{a^2-b^2}\left[\frac{a \operatorname{sen} x}{a+b \cos x} - \frac{b}{\sqrt{a^2-b^2}} \operatorname{arc\,cos}\left(\frac{b+a \cos x}{a+b \cos x}\right)\right]$$

(Bertrand, 115).

Esta función se reduce a $y = p\ (v - r \operatorname{arc\,cos} u)$, cuya diferencial es

$$dy = p\left(d\,v + r\,\frac{d\,u}{\sqrt{1-u^2}}\right) \qquad \text{(A)}$$

$$dv = \frac{(a+b \cos x)\, a \cos x + a \operatorname{sen} x\, b \operatorname{sen} x}{(a+b \cos x)^2}\, d\,x =$$

$$\frac{a^2 \cos x + ab}{(a+b \cos x)^2}\, d\,x \qquad \text{(B)}$$

$$d\,u = \frac{(a+b \cos x)\,(-a \operatorname{sen} x) - (b+a \cos x)\,(-b \operatorname{sen} x)}{(a+b \cos x)^2}\, d\,x$$

$$= \frac{(a^2-b^2) \operatorname{sen} x}{(a+b \cos x)^2}\, d\,x \qquad \text{(C)}$$

$$1-u^2 = 1 - \left(\frac{b+a \cos x}{a+b \cos x}\right)^2 = \frac{(a+b \cos x)^2 - (b+a \cos x)^2}{(a+b \cos x)^2}$$

$$= \frac{(a^2-b^2)\operatorname{sen}^2 x}{(a+b\cos x)^2} \qquad \text{(D)}$$

$$\therefore \frac{du}{\sqrt{1-u^2}} = \frac{(a^2-b^2)\operatorname{sen} x}{(a+b\cos x)^2} \cdot \frac{a+b\cos x}{\operatorname{sen} x\sqrt{a^2-b^2}}$$

$$= -\frac{(a^2-b^2)^{\frac{1}{2}}}{a+b\cos x}\,dx \qquad \text{(E)}$$

Sustituyendo p, r, (B) y (E) en (A),

$$dy = \frac{1}{a^2-b^2}\left(\frac{a^2\cos x+ab}{(a+b\cos x)^2} - \frac{b}{\sqrt{a^2-b^2}} \cdot \frac{(a^2-b^2)^{\frac{1}{2}}}{a+b\cos x}\right)dx$$

$$= \frac{\cos x}{a+b\cos x}\,dx$$

6. $y = x\, e^{\frac{\operatorname{sen} a}{2}\left(x-\frac{1}{x}\right)}$ (Tisserand, 6)

Hacemos $-\frac{1}{2}\operatorname{sen} a = n$, $x - \frac{1}{x} = u$

$$y = x\, e^{nu}$$

$$\therefore dy = e^{nu}\,dx + x\,.\,e^{nu}\,n\,du = e^{nu}(dx + n\,x\,du)$$

$$du = d\left(x - \frac{1}{x}\right) = \left(1 + \frac{1}{x^2}\right)dx = \frac{x^2+1}{x^2}\,dx$$

$$\therefore\ d\,y = e^{-\frac{\text{sen}\ a}{2}\left(x - \frac{1}{x}\right)} \left[1 - \frac{1}{2}\,\text{sen}\ a\left(x + \frac{1}{x}\right)\right] d$$

7. $y = \text{L}\left[1 - \left(1 - e^{-\frac{a}{\text{sen}\ x}}\right)^{\frac{1}{2}}\right]$ (Frenet, 10)

Sea $u = -\dfrac{a}{\text{sen}\ x}$, y $(1 - e^{u})^{\frac{1}{2}} = v \ \therefore\ y = \text{L}\ (1 - v)$

$$\therefore\ d\,y = \frac{d\ (1 - v)}{1 - v} = -\frac{d\,v}{1 - v}$$

$$d\,v = \tfrac{1}{2}\,(1 - e^{u})^{-\frac{1}{2}}\ (-e^{u})\ d\,u$$

$$d\,u = a\left(\frac{-\cos x}{\text{sen}^2\,x}\right) d\,x = \frac{a \cos x}{\text{sen}^2\,x}\,d\,x$$

$$d\,y = \frac{e^{-\frac{a}{\text{sen}\ x}}}{2\left(1 - e^{-\frac{a}{\text{sen}\ x}}\right)\left[1 - \left(1 - e^{-\frac{a}{\text{sen}\ x}}\right)\right]}\ \frac{a \cos x}{\text{sen}^2\,x}\,d\,x$$

8. $u = \left[e^{e^{x}} .\ x^{n} .\ \text{sen}\ x .\ \text{arc cos}\ x\right]^{\frac{1}{x}}$ (Peacock, 40)

Sea $e^{e^{x}} = y$, $x^{n} = z$, $\text{sen}\ x = v$, $\text{arc cos}\ x = t$, $\dfrac{1}{x} = w$

$$u=[y\,z\,v\,t]^{w}$$

Apliquemos L,

$$\text{L}\,u=w\,(\text{L}\,y+\text{L}\,z+\text{L}\,v+\text{L}\,t)$$

$$\therefore \frac{d\,u}{u}=(\text{L}\,y+\text{L}\,z+\text{L}\,v+\text{L}\,t)\,d\,w+w\left(\frac{d\,y}{y}+\frac{d\,z}{z}+\frac{d\,v}{v}+\frac{d\,t}{t}\right)$$

$$=\left(e^{x}+n\,\text{L}\,x+\text{L sen}\,v+\text{L arc cos}\,x\right)\left(-\frac{1}{x^{2}}\right)d\,x$$

$$+\frac{1}{x}\left(e^{x}+\frac{n}{x}+\cot x-\frac{1}{\sqrt{1-x^{2}}\,\text{arc cos}\,x}\right)d\,x$$

Despejando y operando

$$d\,u=\left\{e^{e^{x}}x^{n}\,\text{sen}\,x.\,\text{arc cos}\,x\right\}^{\frac{1}{x}-1}e^{e^{x}}.\,x^{n-2}\,d$$

$$\times\left[\left(e^{x}(x-1)+n\,(1-\text{L}\,x)-\text{L sen}\,x-\text{L arc cos}\,x\right)\text{sen}\,x\,\text{arc cos}\,x+x\left(\cos x\,\text{arc cos}\,x-(1-x^{2})^{-\frac{1}{2}}\,\text{sen}\,x\right)\right]d\,x.$$

9. $$y=\frac{\text{sen}\,(a-b+c)x}{a-b+c}+\frac{\text{sen}\,(a+b-c)\,x}{a+b-c}$$

$$-\frac{\text{sen}\,(a-b-c)\,x}{a-b-c}-\frac{\text{sen}\,(a+b+c)\,x}{a+b+c}.$$ (Brahy, 13)

Representemos los denominadores por m, n, r, s

$$d\left(\frac{\operatorname{sen} m\,x}{m}+\frac{\operatorname{sen} n\,x}{n}-\frac{\operatorname{sen} r\,x}{r}-\frac{\operatorname{sen} s\,x}{s}\right)$$

$$=(\cos m\,x+\cos n\,x-\cos r\,x-\cos s\,x)\,d\,x$$

$$\frac{d\,y}{d\,x}=2\cos\frac{m+n}{2}\,x\,\cos\frac{m-n}{2}\,x-\cos\frac{r+s}{2}\,x\,\cos\frac{r-s}{2}\,x$$

$$=4\cos ax\,\operatorname{sen} bx\,\operatorname{sen} c\,x.$$

340. $y=\mathrm{L}\,(\tfrac{1}{2}\,b+x+\sqrt{a+b\,x+x^2}).$ (Rouché-Lévy, 60)

$$\frac{d\,y}{d\,x}=\frac{1+\dfrac{b+2\,x}{2\sqrt{a\quad b\,x+x^2}}}{\dfrac{b}{2}+x+\sqrt{a+bx+x^2}}=(a+b\,x+x^2)^{-\frac{1}{2}}$$

1. $y=\operatorname{arc\,tg}\dfrac{3\,a^2\,x-x^3}{a\,(a^2-3\,x^2)}.$ (Bertrand, 37)

$$d\,y=\frac{du}{1+u^2};\ \frac{du}{dx}=\frac{a\,(a^2-3\,x^2)\,(3\,a^2-3\,x^2)+(3\,a^2\,x-x^3)\,b\,a\,x}{a^2\,(a^2-3\,x^2)^2}$$

$$\frac{1}{1+u^2}=\frac{a^2\,(a^2-3\,x^2)^2}{(a^2+x^2)^2}\ \therefore\ \frac{dy}{dx}=\frac{3\,a}{a^2+x^2}.$$

* 2. $y=e^{au^2}\operatorname{tg}\dfrac{u^2}{u^2+v^2}$ (Navier, 34)

$$d\,y = e^{au^2 \operatorname{tg} \frac{u^2}{u^2+v^2}}\, d\,a\,u^2 \operatorname{tg} \frac{u^2}{u^2+v^2};$$

$$d\,a\,u^2 \operatorname{tg} \frac{u^2}{u^2+v^2} = a\left[u^2 \sec^2 \frac{u^2}{u^2+v^2}\, d\, \frac{u^2}{u^2+v^2}\right.$$

$$\left. + \operatorname{tg} \frac{u^2}{u^2+v^2}.\ 2\,u\,du\right];\ d\,\frac{u^2}{u^2+v^2} = \frac{2\,uv\,(vdu - udv)}{(u^2+v^2)^2};$$

$$\therefore\ d\,y = e^{au^2 \operatorname{tg} \frac{u^2}{u^2+v^2}}\left[\operatorname{tg} \frac{u^2}{u^2+v^2}\, u\,du + \frac{u^3\,v\,(vdu - udv)}{(u^2+v^2)^2 \cos \frac{u^2}{u^2+v^2}}\right]$$

3. $$y = e^x\,\mathrm{L}\,x$$ (Price, 58)

$$dy = e^x\,\frac{d\,x}{x} + \mathrm{L}\,x.\ e^x\,d\,x = e^x\left(\frac{1}{x} + \mathrm{L}\,x\right) d\,x$$

$$= \frac{e^x}{x}\,(1 + x\,\mathrm{L}\,x)\,d\,x = \frac{e^x}{x}\,[\mathrm{L}\,e + \mathrm{L}\,x^x\,]\,d\,x$$

$$\therefore\ dy = \frac{e^x}{x}\,\mathrm{L}\,e\,x^x\,d\,x$$

4. $$y = \frac{1}{\sqrt{b^2 - a^2}}\,\mathrm{L}\,\frac{\sqrt{b+a} + \sqrt{b-a}\,\operatorname{tg}\frac{1}{2}x}{\sqrt{b+a} - \sqrt{b-a}\,\operatorname{tg}\frac{1}{2}x}.$$ (Pruvost, 439)

$$d\,y = d\,m\,\mathrm{L}\,\frac{n+r\ \mathrm{tg}\,\frac{1}{2}\,x}{n-r\ \mathrm{tg}\,\frac{1}{2}\,x} = m\,\frac{d\,\dfrac{n+r\ \mathrm{tg}\,\frac{1}{2}\,x}{n-r\ \mathrm{tg}\,\frac{1}{2}\,x}}{\dfrac{n+r\ \mathrm{tg}\,\frac{1}{2}\,x}{n-r\ \mathrm{tg}\,\frac{1}{2}\,x}}$$

$$\therefore\ dy = \frac{1}{a+b\cos x}.$$

5. $$y = \mathrm{L}\ \mathrm{tg}\,(\tfrac{1}{4}\,\pi + \tfrac{1}{2}\,x).$$ (Greenhill, 51)

$$d\,y = \frac{\sec^2(\frac{1}{4}\,\pi+\frac{1}{2}\,x).\ \frac{1}{2}\,d\,x}{\mathrm{tg}\,(\frac{1}{4}\,\pi+\frac{1}{2}\,x)} = \frac{d\,x}{2\ \mathrm{sen}\,(\frac{1}{4}\,\pi+\frac{1}{2}\,x)\cos(\frac{1}{4}\,\pi+\frac{1}{2}\,x)}$$

$$= \frac{d\,x}{\mathrm{sen}\,(\frac{1}{2}\,\pi + x)} = \frac{d\,x}{\cos x} = \sec x\,d\,x$$

6. $$y = \frac{1}{\sqrt{r^2\,2+r\,r'\cos x+r'^{\,2}}}$$ (Cournot, 119)

$$d\,y = d\,(r^2+r^2-2\,r\,r'\cos x)^{-\frac{1}{2}} = -\tfrac{1}{2}\,(r^2+r'^{\,2}-2\,r\,r'\cos x)^{-\frac{3}{2}}$$

$$\times\,d\,(r^2+r'^2-2\,r\,r'\cos x) = \frac{-r\,r'\ \mathrm{sen}\,x\,d\,x}{\sqrt{(r^2-2\,r\,r'\cos x + r^2)^3}}.$$

7. $$y = \mathrm{arc\ tg}\left[\sqrt{\frac{a-b}{a+b}}\ \mathrm{tg}\,\frac{x}{2}\right]$$ (Rouché-Lévy, 61)

$$d\,y = \frac{\frac{a-b}{a+b}\sec^2\frac{x}{2}}{1+\frac{a-b}{a+b}\operatorname{tg}^2\frac{x}{2}}\cdot\tfrac{1}{2}\,d\,x = \tfrac{1}{2}\,\frac{\sqrt{a^2-b^2}}{a+b\,\cos\,x}$$

8. $y = \mathrm{L}\,\frac{1+x+x^2}{1-x+x^2} = \mathrm{L}\,(1+x+x^2) - \mathrm{L}\,(1-x+x^2)$ (Lamb, 92)

$$d\,y = \frac{1+2\,x}{1+x+x^2}\,d\,x - \frac{-1+2\,x}{1-x+x^2}\,d\,x$$

* 9. $y = (\operatorname{sen}\,x)^{\mathrm{L}\,x}\cot e^x\,(a+b\,x).$ (Edwards, 49)

$$\mathrm{L}\,y = \mathrm{L}\,x\,\mathrm{L}\operatorname{sen}\,x + \mathrm{L}\cot e^x\,(a+b\,x)$$

$$\frac{d\,y}{d\,x} = (\operatorname{sen}\,x)^{\log x}\cot e^x\,(a+b\,x)\left[\frac{1}{x}\,\mathrm{L}\operatorname{sen}\,x + \cot x\,\mathrm{L}\,x - \right.$$

$$\left. 2\,e^x\,(a+b+b\,x)\operatorname{cosec} 2\,(e^x\,(a+b\,x))\right]$$

350. $z = \operatorname{sen}\left(\tfrac{1}{2}\,k\,\mathrm{L}\,\frac{l+x}{l-x}\right)$ (Boussinesq, 78)

$$d\,y = \left(\cos\tfrac{1}{2}\,k\,\mathrm{L}\,\frac{l+x}{l-x}\right)\tfrac{1}{2}\,k\,d\,\mathrm{L}\,\frac{l+x}{l-x}$$

$$= \frac{k\,l}{l^2-x^2}\cos\left(\tfrac{1}{2}\,k\,\mathrm{L}\,\frac{l+x}{l-x}\right)$$

SEGUNDA SERIE DE EJERCICIOS

351. A. Voss—J. Moik, «Encyclopédie des Sciences Mathématiques», T. II. vol 1. N.º 3.—pág. 253: La derivada de $f\,x$ se designa por $\dot{x}$ (Newton, 1666); por $dy:dx$ (Leibniz, 1675); por $f'(x)$ (Euler, 1765); por y' (Lagrange, 1799); por $D\,f\,x$ (Arbogasto, 1800); por $D_x\,f\,x$ (Cauchy, 1839). p. 255: Las funciones elementales y sus derivadas son:

a) polinomios y sus cuocientes: $D_x\,a=0$, $D_x\,(n\,x)=n$, $D_x\,(x^n)=n\,x^{n-1}$

b) las exponenciales y las logarítmicas: $D_x\,e^x=e^x$, $D_x\,a^x=a^x\log_e a$, $D_x\log_e x=\frac{1}{x}$, $D_x\log_a x=\frac{1}{x}\log_a e=\frac{1}{x\log_e a}$

c) Para las trigonométricas basta saber que

$$\lim_{x=0}\frac{\operatorname{sen} x}{x}=1,\qquad \lim_{x=0}\frac{\operatorname{tg} x}{x}=1$$

D_x sen x; D_x tg x, D_x sec x y líneas complementarias

D_x arc sen x, D_x arc cos x, etc.

$D_x\,s\,h\,x$, etc., y sus inversas.

d) Funciones hiperbólicas.

e) Reglas de Leibniz: $D_x\,(u+v)$, $D_x\,(u+c)$,
$D_x\,(u\,v)$, $D_x\,(c\,u)$,
$D_x\,(u:v)$, $D_x\,(c:u)$.

f) Funciones de funciones: $y=F\,x=F\,(\varphi\,t)=f\,t$

$$\therefore\ D_t\,y=D_x\,y\,.\,D_t\,x$$

g) Funciones inversas: $D_y\,x=\frac{1}{D_x\,y}$.

352. G. H. Hardy. Pure Mathematics. pág. ix; Es más ventajoso escribir arc sen x que $\text{sen}^{-1}x$; y $x \Longrightarrow 0$ que $x=0$. La notación $x \Longrightarrow 0$ es de Leathem y de Bromwich.

p. 114: Como ∞ no es número, $n=\infty$ no tiene significado.

p. 116: En vez de «1 : n es pequeño para grandes valores de n», decimos, con corrección, «1 : $n \Longrightarrow 0$ cuando $n \Longrightarrow \infty$».

p. 192: Reglas de diferenciación: $D\,(f\,x,\ F\,x)=f'\,x+F'\,x$; $D\,(c\,f\,x)=c\,f'\,x$; $D\,(f\,x.\,F\,x)=f\,x\,F'\,x+F\,x\,f'\,x$; $D\,(1:f\,x) = -f'\,x : f^2\,x$; $D\,(f\,x\!:\,F\,x)$; $D\,f\,(x+a)$; $D\,f\,(a\,x)$; $D\,f\,(ax+b)$; funciones inversas y trigonométricas.

p. 195: $y=\text{sen}\,x\cos x=\frac{1}{2}\,\text{sen}\,2\,x \therefore D\,y=-\text{sen}^2 x+\cos^2 x =\cos 2\,x$. Siendo $\text{sen}^2 x+\cos^2 x=1 \therefore D\,(\text{sen}^2 x+\cos^2 x)_n=0$.

En efecto, $D\,(\text{sen}^2 x+\cos^2 x)^n=n\,(\text{sen}^2 x+\cos^2 x)^{n-1}\,(2\,\text{sen}\,x \cos x-2\cos x\,\text{sen}\,x)$.

p. 196: Formas normales (Standard):

A) Polinomios: $D_x\,(a_0\,x^n+a_1\,x^{n-1}+\ldots+a_n)=n\,a_0 x^{n-1} +(n-1)\,x^{n-2}+\ldots.$

B) funciones racionales: $D_x \dfrac{f\,x}{\varphi\,x}=\dfrac{f'\,x}{\varphi\,x}-\dfrac{f\,x\,\varphi'\,x}{\varphi^2\,x}$

C) funciones algebricas: $D_x\,(\sqrt{x}+\sqrt{x+\sqrt{x}})=\dfrac{1}{2\sqrt{x}}$ $\left(1+\dfrac{1+2\sqrt{x}}{2\sqrt{x+\sqrt{x}}}\right)$

D) funciones trascendentes: D_x sen x; D_x arc sen x:

p. 197: $D_x\,(\text{tg}\,x+\sec x)^m+m\,y\sec x$; $D_x\,(\cos ax+i\,\text{sen}\,ax) =a\,i\,y$, $D_x\,(\text{arc sen}\,x+\text{arc cos}\,x)=0$ para $y>\frac{1}{2}\,\pi\,y<0$.

E) funciones de funciones.

353. Duhamel. Calcul Infinitésimal

pág. 222: Las funciones son SIMPLES, FUNCIÓN de FUNCIONES y COMPUESTAS.—Una función es simple cuando un solo signo de operación esta indicado sobre la variable. Las funciones inversas de x^m, a^x, sen x son $\sqrt[m]{y}$, L y, arc sen y.

p. 230: Las funciones simples son $a \pm x$, $a\ x^{\pm 1}$, $x^{\frac{m}{n}}$; a^x, sen x, etc., y las inversas $\sqrt{\ } x$, arc sen x, etc.

Diferenciales simples de: $\log_a x$, a^x, x^m; sen x, cos x; d arc sen $x = d\,x :$ cos $y = \dfrac{d\,x}{\sqrt{1-x^2}}$

354. B. Price. Differential Calculus.

pág. 47: Regla 1.ª: $d f x = f' x\, d\, x$; 2.ª: $d\, c\, f\, x = c\, f'\, x\, d\, x$; 3.ª: $d\,(f\ x + F\ x)$; 4.ª: $d\ (f\ x.\ F\ x)$; 5.ª: $d\ (f\ x\colon\ F\ x)$; 6.ª: $d\,(x^n)$; 7.ª: $d\ (x^x)$; 8.ª: d ($\log_a x$); 9.ª: d(sen x); 10.ª: d (arc sen x).

355. Ch. de Comberousse: Algèbre Supérieure.

pág. 437: La derivada se representa por $f'(x)$, $D\,f\,x$, $D\,y$ ó $D_x\,y$.

p. 454: funciones simples: $a \pm x$, $a x^{\pm 1}$, $x^{\frac{m}{n}}$, a^x, $\log_a x$, sen x y arc sen x. función de función: $y = a\,\mathrm{L}\,x$; compuesta; inversa.

p. 464: fórmulas: $D\,(u+v-t)$, $D\,(a \pm x)$, $D\,(uv)$, $D\,(u:v,)$, $D\,(u^u)$, D ($\log_a x$), $D\,(a^x)$, D (sen x), D (arc sen x).

(*Continuará*).